A Martingale Lattice

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Abstract

A gambler starts with \$1. He then plays a sequence of gambles.

1. We pose the question "What does a gambler lose when he gambles?" For a fair gamble, the gambler loses position in the Lorenz ordering after every gamble.
2. The gambler may not want to place too much money on the table. The gambler wants a strategy that minimizes the money placed on the table while achieving a target distribution. Towards that end, private randomness is helpful
3. A real casino offers a set of unfair gambles. Even though a casino offers a limited set of games, a richer set of games is available by strategically combining the menu items, unless the original game menu is price stable, which we characterize for the binary case with a simple example.

Some randomness is free

Suppose the casino offers binary fair bets:

$$X \to \begin{cases} 2X, & \text{prob}\,\frac{1}{2} \\ 0, & \text{prob}\,\frac{1}{2} \end{cases}$$

Suppose a gambler starts with \$1.
Suppose that he wants to achieve a uniform distribution $\text{unif}(0,2)$ on his wealth after gambling.
There is more than one possible gambling strategy to achieve this:

Method 1:

- Bet \$$\frac{1}{2}$, then bet \$$\frac{1}{4}$, then \$$\frac{1}{8}$, ...
- After infinite bets the distribution of wealth is $\text{unif}(0,2)$.

In this method, the gambler must place on the casino's table a total of:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1$$

Method 2:

- Generate $Y \sim \text{unif}(0,1)$.
- Bet \$Y.

In this method, the gambler places on the table a total amount

$$E(Y) = \frac{1}{2} < 1$$

Thus the use of private randomness has an advantage to using the casino for supplying the requisite randomness.

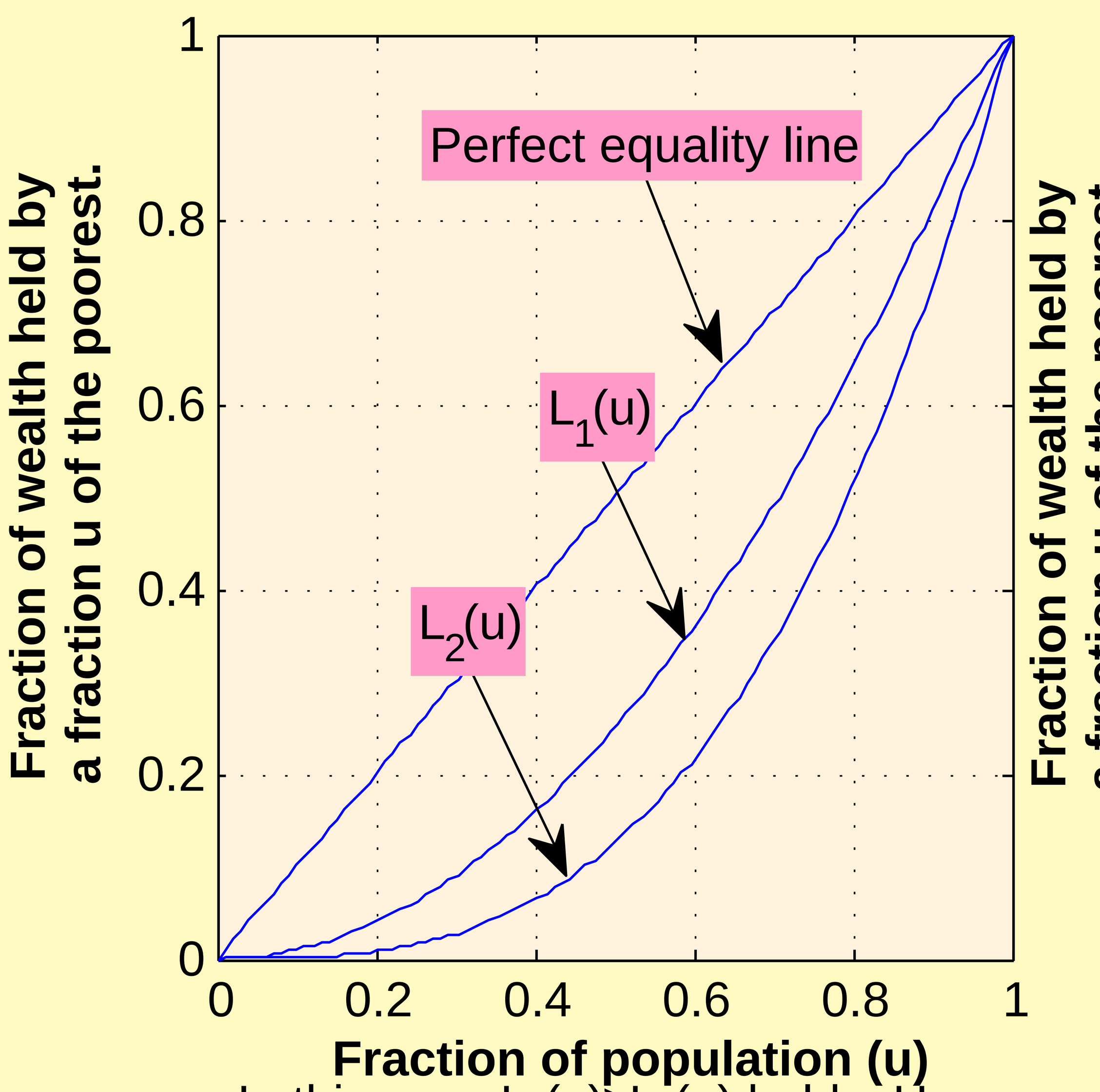
References

[1] Strassen (1965): The existence of probability measures with given marginalsls. http://www.jstor.org/stable/pdfplus/2238148.pdf

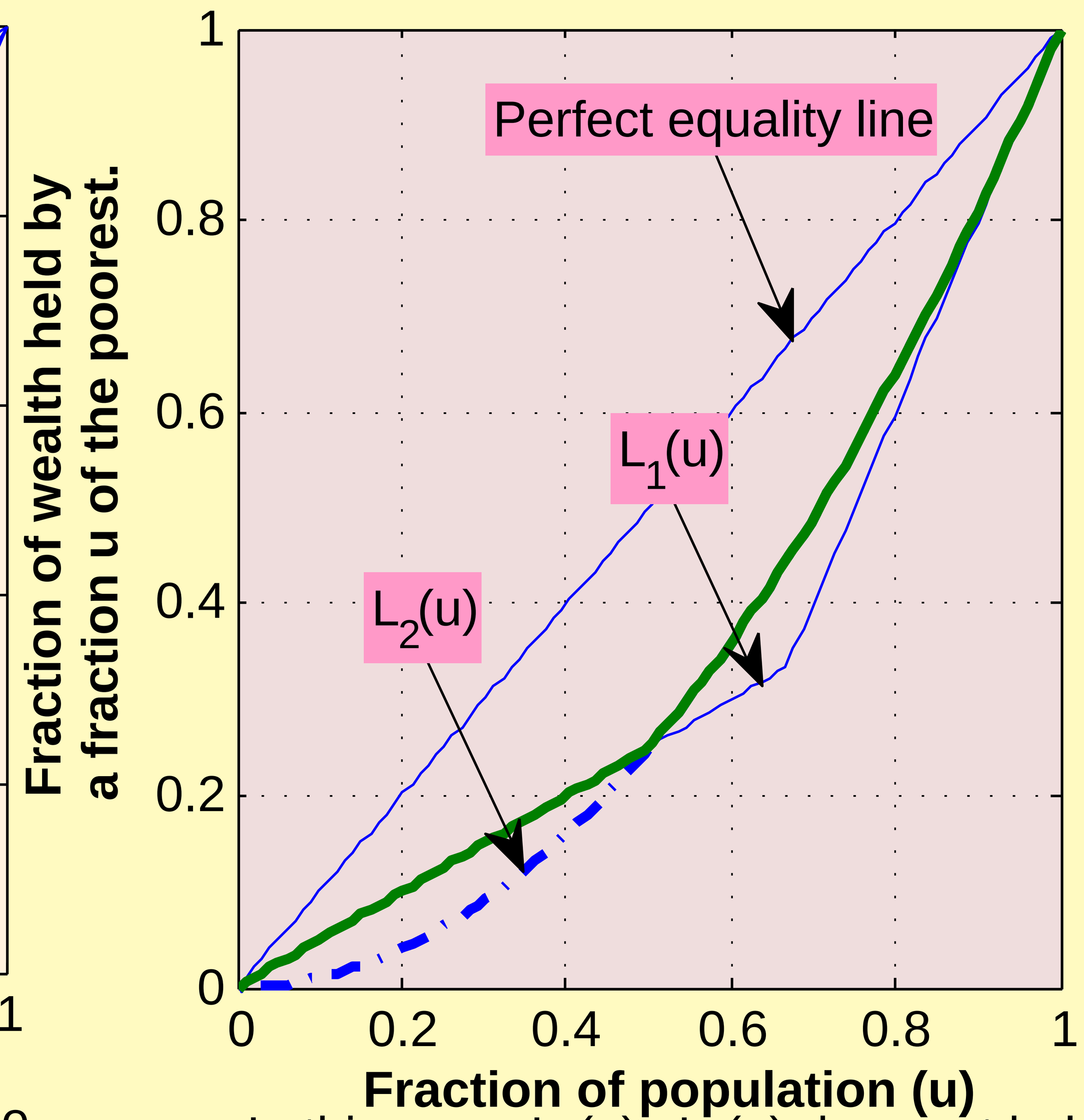
[2] Foster and Vohra (1996): Calibrated Learning and Correlated Equilibrium. http://eprints.kfupm.edu.sa/29140/1/29140.pdf

Gambling increases inequality

Gambling increases inequality: Suppose there is a nation of gamblers each starting with \$1 (perfect equality, represented by the line y=x on the Lorenz curve). Suppose each gambler gambles using the same strategy on i.i.d casinos. Then the lorenz curve, representing the inequality in wealth distribution, always goes downstream. Further, given a distribution, it is always possible to find a sequence of gambles that lead to a more unequal distribution (downstream in the Lorenz curve).



In this case $L_1(u) \geq L_2(u)$ holds. Hence, there exists a sequence of gambles that start at distribution X_1 and go to X_2.



In this case, $L_1(u) \geq L_2(u)$ does not hold.
The green line shows the "best" distribution from which both gambles are achievable.

- Let F_1 and F_2 correspond to the laws of the random variables X_1 and X_2 respectively.
- Let the corresponding Lorenz Curves be denoted L_1 and L_2 respectively.
- **Theorem:** There is a sequence of fair binary gambles that starts from distribution X_1 and ends up with a distribution X_2 iff

$$L_1(u) \geq L_2(u) \quad \forall u \in [0,1]. \tag{1}$$

- **Achievability:**
  - Let $U_1 = F_1(X_1)$. Also, let $X_1' = F_2^{-1}(U_1)$.
  - If $X_1' = X_1$ then do nothing.
  - If $X_1' < X_1$, find the minimum $u > U_1$ such that $L_1(u) - L_2(u) = L_1(U_1) - L_2(U_2)$ and label it U_1''.
  - Let $X_1'' = F_2^{-1}(U_1'')$.
  - Gamble to the two points X_1' and X_1'' with the proper weights such that it is a fair gamble.
  - In the opposite case where $X_1' > X_1$, do the same thing using the maximum $u < U_1$ meeting the named condition.

Fair prices of unfair gambles

Let $f : [0,1] \to [0,1]$ be a binary gamble menu offered by a casino, that is, for each $x \in [0,1]$,

$$x \to \begin{cases} 1, & \text{prob} \quad f(x) \\ 0, & \text{prob} \quad 1 - f(x) \end{cases}$$

The gambling menu is fair if $f(x) = x$.
Defn: We say that a gambling menu is **stable** if it cannot be gamed to produce a more attractive menu.
Necessary and sufficient condition for a stable gambling menu:

$$\forall\, a, b, \theta \in [0,1] \text{ where } b > a$$
$$f(a + (b-a)\theta) \geq f(a) + (f(b) - f(a)) f(\theta).$$

Example:
Suppose $f(x) = px,\ p < 1$, that is, the casino charges a fixed fraction p in expected value. This is not price stable.
A clever gambler would use an infinite sequence of very low probability and low cost gambles to achieve $f(x) = 1 - (1-x)^p$. This is price stable!